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Independent Domination in Line Graphs

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Abstract - For any graph G, the line graph $L(G)=H$ is the intersection graph. Thus the vertices of $L(G)$ are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are. A dominating set D is called independent dominating set of $L(G)$, if D is also independent. The independent domination number of $L(G)$ denoted by $i(L(G))$, equals $\min\{|D|; D$ is an independent dominating set of $L(G)\}$. A domatic partition of $L(G)$ is a partition of $V(L(G))$, all of whose classes are dominating sets in $L(G)$. The maximum number of classes of a domatic partition of $L(G)$ is called the domatic number of $L(G)$ and denoted by $d(L(G))$. In this paper many bounds on $i(L(G))$ were obtained in terms of elements of G, but not in terms of elements of $L(G)$. Further we develop its relationship with other different domination parameters. Also we introduce the concept of domatic number in $L(G)$.

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Key words: Domatic number, Domination number, Graph, Independent dominating set, Independent domination number, Line graph and Odd graph.

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1 INTRODUCTION:

In this paper, we follow the notations of [1]. All the graphs considered here are simple, finite, non-trivial, undirected, and connected. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edge of a graph G respectively. In general we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v. The notation $\beta_0(G)$ $(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G.

Let $\deg(v)$ is the degree of vertex v and a vertex of degree one is called an end vertex and its neighbor is called a support vertex.

As usual $\delta(G)$ $(\Delta(G))$ is the minimum (maximum) degree. The degree of an edge $e=uv$ of G is defined by $\deg e=\deg u+\deg v-2$ and $\delta'(G)$ $(\Delta'(G))$ is the minimum (maximum) degree among the edges of G.

A spider is a tree with the property that the removal of all end paths of length two of T results in an isolated vertex, called the head of the spider

Let $k\geq 2$ be an integer. The odd graph O_k is the graph whose vertex set is V and in which two vertices are adjacent if and only if they are disjoint as sets.

A line graph $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A set $D\subseteq V$ is said to be dominating set of G, if every vertex not in D is adjacent to a vertex in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set.

A set D is an independent dominating set of G, if D is also independent. The independent

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domination number of G, denoted by $i(G)$ is the minimum cardinality of an independent dominating set. The concept of domination is now well studied in graph theory (see [2], [3]).

Analogously, a set $D' \subseteq V(L(G))$ is said to be dominating set of $L(G)$, if every vertex not in D is adjacent to a vertex in D. The domination number of $L(G)$, denoted by $\gamma(L(G))$ is the minimum cardinality of a dominating set in $L(G)$.

A dominating set D' of $L(G)$ is said to be independent dominating set if D' is also independent. The independent domination number of $L(G)$, denoted by $i(L(G))$ is the minimum cardinality of an independent dominating set in $L(G)$.

A domatic partition in $L(G)$ is a partition of $V(L(G))$, all of whose classes are dominating sets in $L(G)$. The maximum number of classes of a domatic partition of $L(G)$ is called the domatic number of $L(G)$ and is denoted by $d(L(G))$. The concept of domatic number in G was introduced by Cockeyne et.al, [4].

In this paper, many bounds on $i(L(G))$ were obtained. Also their relationships with other domination parameters were obtained. Further, we introduce the concept of domatic number of $L(G)$, exact values of domatic number were obtained for some standard graphs. Also bound for domatic number is also obtained.

2 RESULTS:

Initially we list out independent domination number of $L(G)$ for some standard graphs.

Theorem 1:

a. $i(L(C_p)) = \lceil p/3 \rceil$.

b. $i(L(K_{1,n})) = 1$.

c. $i(L(K_{m,n})) = n$ *for* $m \geq n$.

d. $i(L(K_p)) = \lfloor p/2 \rfloor$.

e. $i(L(W_p)) = \left\lceil \frac{(p+2)}{3} \right\rceil$.

The following Theorem relates domination and independent domination in $L(G)$.

Theorem 2: *For any connected* (p,q) *– graph,* $\gamma(L(G)) + i(L(G)) \leq q$. *Equality holds if* $G \cong C_4$.

Proof: Suppose $D = \{v_1, v_2, v_3, ..., v_n\} \subseteq V(L(G))$ be the set of vertices which covers all the vertices in $L(G)$. Then D is a minimal γ - set of $L(G)$. Further, if the sub graph $\langle D \rangle$ contains the set of vertices $v_i, 1 \leq i \leq n$, such that $\deg(v_i) = 0$. Then D itself is an independent dominating set of $L(G)$. Otherwise, $S = D' \cup I$, where $D' \subseteq D$ and $I \subseteq V(L(G)) - D$ forms a minimal independent dominating set of $L(G)$.Since $V(L(G)) = E(G)$, it follows that $|D| \cup |S| \leq q$. Therefore, $\gamma(L(G)) + i(L(G)) \leq q$.

For equality, suppose $G \cong C_4$, then in this case $|D| = |S| = 2 = q/2$. Clearly, it follows that, $\gamma(L(G)) + i(L(G)) = q$.

In the following Theorems we give the upper bounds for independent domination number of $L(G)$.

Theorem 3: *If every support vertex of tree is adjacent to at least one end edge, then* $i(L(T)) \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$, *where* m *is the number of end edges in* T. *Equality holds for star* $K_{1,p-1}$.

Proof: Let $F = \{e_1, e_2, e_3, ..., e_n\}$ be the set of all end edges in T such that $|F| = m$. Now without loss of generality, since $V(L(T)) = E(T)$, let $S = F' \cup H$, where $F' \subseteq F$ and $H \subseteq V(L(T)) - F$, such that $H \notin N[F]$ be the minimal set of vertices which covers all the vertices in $L(T)$. Clearly set of the vertices of a sub graph $\langle S \rangle$ is independent, then by the above argument S is a minimal independent dominating set of $L(T)$. Clearly it follows that, $|S| \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$.

Therefore, $i(L(T)) \leq \left\lceil \frac{q-m}{2} \right\rceil + 1$.

Suppose $T \cong K_{1,p-1}$. Then in this case, $q = m$. Since $L(T) = K_p$ and $|S| = 1$, it follows that $i(L(T)) = \left\lceil \frac{q-m}{2} \right\rceil + 1$.

Theorem 4: *For any connected graph* G, $i(L(G)) \le \beta_1(G)$.

Proof: Suppose $J = \{e_1, e_2, e_3, ..., e_n\}$ be the maximum set of edges in G such that $N(e_i) \cap N(e_j) = \phi$ for $1 \le i \le j \le n$. Then j forms maximal independent set of edges with $|j| = \beta_1(G)$. Since $V(L(G)) = E(G)$, there exists an independent set $D = J' \cup H$, where $J' \subseteq J$ and $H \subseteq V(L(G)) - J$ such that $H \notin N[J]$, which covers all the vertices in $L(G)$. Clearly, D forms a minimal independent dominating set of $L(G)$ and it follows that $|D| \le |J|$. Therefore, $i(L(G)) \le \beta_1(G)$.

Theorem 5: *For any connected* (p,q) *– graph* G, $i(L(G)) \le q - \Delta'(G)$.

Proof: Suppose $C = \{v_1, v_2, v_3, ..., v_k\}$ be the set of all non end vertices in G. Then there exists at least one vertex $v \in C$ which is incident with at least one edge $e \in \Delta'(G)$ in G. Now without loss of generality in $L(G)$, suppose $H = \{v_1, v_2, v_3, ..., v_n\}$ be the set of all end vertices in $L(G)$ and if $V(L(G)) - H = I$. Then there exists a subset $D \subseteq I$ in $L(G)$ such that the sub graph $\langle D \rangle$ is independent. Clearly, D is an i - set of $L(G)$. It follows that, $|D| \le q - \Delta'(G)$ and hence $i(L(G)) \le q - \Delta'(G)$.

Corollary 1: *For any tree* T *containing an edge* $e \in \Delta'(T)$, $i(L(T) = q - \Delta'(T)$ *if and only if* $diam(T) \le 3$.

Theorem 6: *For any connected* (p,q) *– graph* G, $i(L(G)) = 1$. *If and only if* G *contains an edge* $e \in E(G)$ *such that* $\deg e = q - 1$.

Proof: Assume that $\deg e < q - 1$. Now in $L(G)$, $V(L(G)) = E(G)$, suppose $F = \{v_1, v_2, ..., v_n\}$ be the set of all end vertices in $L(G)$. Then there exists a vertex set $D = \{v_1, v_2, v_3, ..., v_k\} \subseteq V(L(G)) - F$, which covers all the vertices in $L(G)$. Further, if the sub graph $\langle D \rangle$ is independent, then D forms a minimal independent dominating set of $L(G)$ with $|D| \ge 2$, a contradiction.

Suppose $\deg e = q - 1$. Then in $L(G)$, since $V(L(G)) = E(G)$, there exists a vertex $v \in V(L(G))$ which covers all vertices in $L(G)$ and $v \in D$. Clearly, D itself is a minimal independent dominating set of $L(G)$. Therefore, $|D| = 1$ and hence $i(L(G)) = 1$.

Theorem 7: *For any tree* T, $i(L(T)) \le (p-1)/2$. *Equality holds if and only if* T *is isomorphic to a spider.*

Proof: Let $F' = \{v_1, v_2, v_3, ..., v_i\} \subseteq V(L(G))$ be the set of all vertices with $F' \in N(F)$, where F is the set of end vertices of $L(T)$. Suppose $H \subseteq V(L(T)) - F'$ and $H \notin N(F)$. Then $D = I \cup F'$ where $I \subseteq H$, covers all the vertices in $L(T)$. Further, if the sub graph $I \cup F'$ is independent, then $D = I \cup F'$ forms a minimal independent dominating set of $L(T)$. Clearly, it follows that $|I \cup F'| \le (p-1)/2$. Therefore, $i(L(T)) \le (p-1)/2$. Suppose T is isomorphic to spider. Then in this case $|D| = |F|$ or $|D| = \frac{q}{2}$. Since, for any tree T, $q = p - 1$ and each p in T is odd, it follows that, $i(L(T)) \le (p-1)/2$.

In the following Theorem, we list out the domatic number for some standard graphs.

Theorem 8:

I. *For any complete graph* K_p,

$d\ \ L\ \ K_p\ \ = p-1$, *if* p *is even.*

= p, *if* p *is odd.*

II. *For any complete bipartite graph* K_{p_1,p_2} *with* $p_1 + p_2 = p$ *vertices,*

$d\ \ L\ \ K_{p_1,p_2}\ \ = max\ \ p_1, p_2$.

III. *For any cycle* C_p*,*

$d\ \ L\ \ C_p\ \ = 3$, *if* p *is divisible by 3.*

= 2, otherwise.

IV. *For any path* P_p*,*

$d\ \ L\ \ P_p\ \ = 2$.

V. *For any star* $K_{1,n}$ *with* $1+n = p$ *vertices,*

$d\ \ L\ \ K_{1,n}\ \ = n$.

Proposition 1: *The domatic number of a line graph of* O_k *where* O_k *is an odd graph is equal to* $2k-1$.

Proof: The edge domatic number of a graph is evidently equal to the domatic number of its line graph [1]. The degree of each vertex of line graph of O_k is $2k-2$ and this implies that its domatic number is at most $2k-1$. Therefore $d\ \ L\ \ O_k\ \ = 2k-1$.

Theorem 9: *For any connected graph* G, $\delta(G) \le d(L(G)) \le \delta_e(G)+1$, *where* $d\ \ L\ \ G$ *is the domatic number of* $L\ \ G$, $\delta\ \ G$ *is the minimum degree of a vertex of* G *and* $\delta_e(G)$ *is the minimum degree of an edge of* G .

Proof: The number $\delta_e(G)$ is equal to the minimum degree of an edge of the line-graph of G. According to [1], the domatic number of this line graph cannot be greater than $\delta_e(G)+1$.

Now we shall prove the first inequality $\delta(G) \le d(L(G))$ by the method of induction. If the degree of each vertex of $L\ \ G$ is greater than or equal to k, where k is an arbitrary positive integer, then there exists a domatic partition of $L\ \ G$ with k classes. For $k=1$ the result is true. The required partition consists of one class equal to the whole $V\ \ L\ \ G$ which is evidently a dominating set in $L\ \ G$. Now let $k_0 \ge 2$ and suppose that the result is true for $k = k_0 - 1$. Consider a line graph $L\ \ G$ in which the degree of each vertex is at least k_0. Let $V_0\ \ L\ \ G$ be a maximal (with respect to the set inclusion) independent set of vertices of $L\ \ G$. This set is dominating; otherwise a vertex could be added to it without violating the independence, which would be a contradiction with the maximality of $V_0\ \ L\ \ G$. Let $L\ \ G_0$ be a line graph obtained from $L\ \ G$ by deleting all vertices of $V_0\ \ L\ \ G$. Each vertex of $L\ \ G$ is incident at most with one vertex of V_0, therefore each vertex of $L\ \ G_0$ has the degree at least $k_0 - 1$. According to the hypothesis, there exists a vertex domatic partition P of $L\ \ G_0$ with $k_0 - 1$ classes. Therefore $P \bigcup V_0\ \ L\ \ G$ is a vertex domatic partition of $L\ \ G$ with k_0 classes and it implies that $d(L(G)) \ge \delta(G)$.

For equality, we have the following Cases.

If G is a cycle C_p with p is divisible by 3. Then $d(L(G)) = \delta_c(G)+1$.

If G is a cycle C_p with p is not divisible by 3. Then $d(L(G)) = \delta(G)$.

Finally, we give the following Characterization.

Theorem 10: *Let* T *be a tree, let* $\delta\ \ L\ \ T$ *be the minimum degree of a vertex of* $L(T)$. *Then* $d(L(T)) = \delta(L(T))+1$.

Proof: Let us have the colors $1, 2, ..., \delta(L(T))+1$; we shall color the vertices of $L(T)$. First we choose a terminal vertex v_0 of $L(T)$ and color it by 1. Now let us have a vertex v of $L(T)$ with end vertices u, w; suppose that all vertices incident with w are already colored. Moreover, if the number of these vertices is less than $\delta(L(T))+1$, we suppose that they are colored by pair wise different colors. In the opposite case we suppose that all colors $1, 2, ..., \delta(L(T))+1$ occur among the colors of these vertices. Now we shall color the vertices adjacent to u and distinct from v. We color them in the following way. If there are colors by which no vertex adjacent to v is colored, we use all of them. (This must be always possible according to the assumption). If the number of vertices to be colored is less than $\delta(L(T))+1$, we color them by pair wise distinct colors; in the opposite case we color them by using all the colors $1, 2, ..., \delta(L(T))+1$ (it may form the color class). The result is a coloring of vertices of $L(T)$ by the colors $1, 2, ..., \delta(L(T))+1$ with the property that each vertex is adjacent to vertices of all colors different from its own. If A_i for $i = 1, 2, ..., \delta(L(T))+1$, is the set of all vertices of T colored by i, then the sets $A_1, A_2, ..., A_{\delta(L(T))+1}$ form a domatic partition of $L(T)$ with $\delta(L(T))+1$ classes and $d(L(T)) \geq \delta(L(T))+1$. According to Theorem 9, it cannot be greater, therefore $d(L(T)) = \delta(L(T))+1$.

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